SOLUTION OF HEAT-TRANSFER PROBLEMS IN A LAYER

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UDC 536.244:532.546.2

A method is presented for solving the heat-transfer problem in a layer of a plate, cylinder, or sphere. Exact and approximate solutions and an algorithm for computer calculations are presented. The approximate solution is expressed in terms of tabulated functions and is convenient for engineering calculations.

Aksel'rud [1] gives a generalized solution of the heat-transfer problem in a fixed layer of a massive plate, cylinder, or sphere for simple boundary conditions — constant temperature of the moving medium (gas) in front of the layer and a constant initial temperature over the whole volume of the layer. An expression for the temperature of the gas is obtained in quadratures using Thomson functions.

Shklyar et al. [2] solve the problem of the heating of a plate by a flow of gas with more general boundary conditions: the temperature of the gas in front of the layer is a function of time, and the initial temperature of the material is a function of the coordinates along the thickness and lengthwise of the layer. The final result is also obtained in quadratures. Both solutions are complicated for practical use.

We describe below a method for the exact solution of the problem for a layer of bodies of various shapes for boundary conditions of the type [2], and also give an approximate solution of this problem in a form convenient for engineering calculations.

The well-known approximate solutions of the problem [3, 4] are based on the laws of heating a layer of bodies of perfect thermal conductivity, taking account of the coefficient of massiveness. The approximate solution presented is obtained from the exact solution by retaining only the first term of the series. Heat transfer along the layer of material is neglected. The problem posed is described by the following equations:

$$\frac{\partial t(r,\ \mathrm{Fo},\ \mathrm{St})}{\partial \mathrm{Fo}} = \frac{\partial^2 t(r,\ \mathrm{Fo},\ \mathrm{St})}{\partial^2 r} + \frac{\nu}{r} \cdot \frac{\partial t(r,\ \mathrm{Fo},\ \mathrm{St})}{\partial r}, \tag{1}$$

where $\nu = 0, 1, 2$ respectively for a plate, cylinder, and sphere; $\mathrm{St} = \alpha\ F/Vc$;

$$t(1,\ \mathrm{Fo},\ \mathrm{St}) - T(\mathrm{Fo},\ \mathrm{St}) = \frac{\partial T(\mathrm{Fo},\ \mathrm{St})}{\partial \mathrm{St}}, \tag{2}$$

$$\left.\frac{\partial t}{\partial r}\right|_{r=1} = \mathrm{Bi}[T(\mathrm{Fo},\ \mathrm{St}) - t(1,\ \mathrm{Fo},\ \mathrm{St})], \tag{3}$$

$$\left.\frac{\partial t}{\partial r}\right|_{r=0} = 0, \tag{4}$$

$$\mathrm{St} = 0,\ T = \varphi(\mathrm{Fo}), \tag{5}$$

$$\mathrm{Fo} = 0,\ t = f(r,\ \mathrm{St}). \tag{6}$$

The heat-balance equation of the gas (2) can be related to the boundary conditions of the heat-conduction problem since it expresses the variation of the temperature of the medium T(Fo, St) with time and along the length of the layer. Using Duhamel's theorem we obtain the solution of the more general problem (1), (3)-(6) for an arbitrary function T(Fo, St). This solution has the form

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Dnepropetrovsk Metallurgical Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 9, pp. 497-503, September, 1974. Original article submitted June 6, 1972.

$$t(r,\ \text{Fo},\ \text{St}) = T(\text{Fo},\ \text{St}) - T(0,\ \text{St}) \sum_{n=1}^{\infty} A(\mu_n) u(r,\ \mu_n)$$

$$\times \exp(-\mu_n^2 \text{Fo}) - \int_0^{\text{Fo}} \frac{\partial T(\omega,\ \text{St})}{\partial \omega} \sum_{n=1}^{\infty} A(\mu_n) u(r,\ \mu_n)$$

$$\times \exp[-\mu_n^2(\text{Fo} - \omega)]\, d\omega + \sum_{n=1}^{\infty} A'(\mu_n) u'(r,\ \mu_n,\ \text{St}) \exp(-\mu_n^2 \text{Fo}), \tag{7}$$

where $A(\mu_n)$, $u(r, \mu_n)$, $A'(\mu_n)$, and $u'(r, \mu_n,\ \text{St})$ are components of the known solution of Eq. (1) for a plate, cylinder, and sphere for a constant temperature of the medium [5].

If the function T(Fo, St) is chosen so that condition (2) is satisfied, (7) gives the solution of the problem posed. To determine T(Fo, St) we find an expression for the temperature of the surface of the material t(1, Fo, St) from (7) and substitute it into (2). This gives an integral equation for the required function T(Fo, St):

$$\frac{\partial T}{\partial \text{St}} = -T(0,\ \text{St}) \sum_{n=1}^{\infty} A(\mu_n) u(1,\ \mu_n) \exp(-\mu_n^2 \text{Fo})$$

$$-\int_0^{\text{Fo}} \frac{\partial T}{\partial \omega} \sum_{n=1}^{\infty} A(\mu_n) u(1,\ \mu_n) \exp[-\mu_n^2(\text{Fo} - \omega)]\, d\omega + \sum_{n=1}^{\infty} A'(\mu_n) u'(1,\ \mu_n,\ \text{St}) \exp(-\mu_n^2 \text{Fo}). \tag{8}$$

After taking the Laplace transform of the last equation with respect to Fo we have

$$\frac{\partial \overline{T}(p,\ \text{St})}{\partial \text{St}} + \sum_{n=1}^{\infty} A(\mu_n) u(1,\ \mu_n) \frac{p}{p + \mu_n^2} \overline{T}(p,\ \text{St}) = \sum_{n=1}^{\infty} A'(\mu_n) u'(1,\ \mu_n,\ \text{St}) \frac{1}{p + \mu_n^2}. \tag{9}$$

Solving Eq. (9) and taking the inverse transform as in [2] we find for the temperature of the gas

$$T(\text{Fo},\ \text{St}) = \varphi(\text{Fo}) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{1}{\beta} \exp(-\text{St}\, N) \sin(-\text{St}\, M)\, d\beta \right\}$$

$$+ \frac{1}{\pi} \int_0^{\text{Fo}} \varphi(\omega) \left\{ \int_0^{\infty} \exp(-\text{St}\, N) \cos[\beta(\text{Fo} - \omega) - \text{St}\, M]\, d\beta \right\} d\omega$$

$$+ \frac{1}{\pi} \int_0^{\infty} d\beta \int_0^{\text{St}} \{C(\beta,\ \zeta) \cos[\beta \text{Fo} + (\zeta - \text{St}) M] + D(\beta,\ \zeta) \sin[\beta \text{Fo}$$

$$+ (\zeta - \text{St}) M]\} \exp[(\zeta - \text{St}) N]\, d\zeta, \tag{10}$$

where

$$M = \sum_{n=1}^{\infty} A(\mu_n) u(1,\ \mu_n) \frac{\beta \mu_n^2}{\mu_n^4 + \beta^2};$$

$$N = \sum_{n=1}^{\infty} A(\mu_n) u(1,\ \mu_n) \frac{\beta^2}{\mu_n^4 + \beta^2};$$

$$C(\beta,\ \zeta) = \sum_{n=1}^{\infty} A'(\mu_n) u'(1,\ \mu_n,\ \zeta) \frac{\mu_n^2}{\mu_n^4 + \beta^2};$$

$$D(\beta,\ \zeta) = \sum_{n=1}^{\infty} A'(\mu_n) u'(1,\ \mu_n,\ \zeta) \frac{\beta}{\mu_n^4 + \beta^2}.$$

The solution for a layer of plane bodies found in [2] differs from the generalized solution (10), but simple transformations show that they are identical.

TABLE 1. Values of the Temperature of a Gas T_5 Calculated by Summing Five Terms of the Series and T_1 by Retaining Only the First Term

Fo	St=0,5		St=1,0		St=2,0		St=5,0		St=10,0	
	T_5	T_1	T_5	T_1	T_5	T_1	T_5	T_1	T_5	T_1
					Bi=1					
0,1	0,70	0,71	0,49	0,51	0,24	0,26	0,04	0,03	0,025	0,002
0,5	0,78	0,78	0,60	0,60	0,35	0,35	0,08	0,07	0,027	0,004
2,0	0,91	0,91	0,82	0,82	0,63	0,63	0,23	0,23	0,05	0,03
5,0	0,99	0,99	0,96	0,96	0,90	0,90	0,58	0,58	0,18	0,15
10,0	1,00	1,00	1,00	1,00	0,99	0,99	0,90	0,90	0,56	0,56
					Bi=5					
0,1	0,86	0,88	0,73	0,77	0,52	0,59	0,17	0,26	0,034	0,068
0,2	0,89	0,89	0,79	0,80	0,60	0,64	0,24	0,32	0,055	0,094
0,5	0,93	0,93	0,87	0,87	0,74	0,75	0,41	0,46	0,13	0,18
1,0	0,97	0,97	0,94	0,94	0,86	0,87	0,61	0,65	0,28	0,34
3,0	1,00	1,00	1,00	1,00	0,99	0,99	0,94	0,95	0,78	0,81
					Bi=10					
0,1	0,92	0,93	0,83	0,86	0,68	0,75	0,32	0,48	0,08	0,23
0,2	0,94	0,94	0,88	0,89	0,76	0,79	0,46	0,54	0,16	0,29
0,5	0,97	0,97	0,94	0,94	0,87	0,87	0,66	0,69	0,36	0,45
1,0	0,99	0,99	0,97	0,98	0,94	0,95	0,82	0,84	0,59	0,66

The expression for the temperature of the materials is obtained by substituting (10) and its derivative with respect to Fo into (7). The final result in the form (10) is unsuitable for calculations, even for simple boundary conditions.

A computer algorithm based on Eqs. (7) and (8) has been developed for obtaining accurate numerical results.

Transforming the integral in (7) and (8) by the integration formula

$$\int_{\mathrm{Fo}_{k-1}}^{\mathrm{Fo}_k} \exp(\mu_n^2\omega)\,\frac{\partial T}{\partial\omega}\,d\omega \approx \exp(\mu_n^2\,\mathrm{Fo}_k)(T_k - T_{k-1}) \tag{11}$$

and going to finite differences in time and lengthwise of the layer we obtain the following recurrence relations

$$T_{m+1,k} = T_{m,k} - \Delta\,\mathrm{St}\left\{\sum_{n=1}^{\infty} A(\mu_n)\,u(1,\ \mu_n)\,[\exp(-\mu_n^2 k\Delta\,\mathrm{Fo})\,T(0,\ (m+1)\,\Delta\,\mathrm{St}) + L_{m,k,n}] + \sum_{n=1}^{\infty} A'(\mu_n)\,u'\,[1,\ \mu_n,\ (m+1)\,\Delta\,\mathrm{St}]\exp(-\mu_n^2 k\Delta\,\mathrm{Fo})\right\}, \tag{12}$$

$$t_{m+1,k} = T_{m+1,k} - \sum_{n=1}^{\infty} A(\mu_n)\,u(r,\ \mu_n)\{\exp(-\mu_n^2 k\Delta\,\mathrm{Fo})\,T\,[0,\ (m+1)\,\Delta\,\mathrm{St}] + L_{m,k,n}\} + \sum_{n=1}^{\infty} A'(\mu_n)\,u'\,[r,\ \mu_n,\ (m+1)\,\Delta\,\mathrm{St}]\exp(-\mu_n^2 k\Delta\,\mathrm{Fo}), \tag{13}$$

where

$$L_{m,k,n} = L_{m,k-1,n}\exp(-\mu_n^2\Delta\,\mathrm{Fo}) + T_{m,k} - T_{m,k-1}.$$

For $k=0$, $L_{m,0,n} = 0$; $k = 0, 1, 2, 3, \ldots$; $m = 0, 1, 2, 3, \ldots$. Calculations with algorithm (12)-(13) do not require a large expenditure of machine time. The heating of a plate by the flow of a gas was calculated on a Minsk-22 computer for $\varphi(\mathrm{Fo}) = 1$ and $f(r, \mathrm{St}) = 0$. The computational formulas for this case have the form

$$T_{m+1,k} = T_{m,k} - \Delta\,\mathrm{St}\sum_{n=1}^{\infty}\frac{2\sin\mu_n\cos\mu_n}{\mu_n + \sin\mu_n\cos\mu_n}\,L'_{m,k,n}, \tag{14}$$

$$t_{\mathrm{sur},m+1,k} = T_{m+1,k} - \sum_{n=1}^{\infty} \frac{2\sin\mu_n \cos\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} L'_{m,k,n}, \tag{15}$$

$$t_{\mathrm{ax},m+1,k} = T_{m+1,k} - \sum_{n=1}^{\infty} \frac{2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} L'_{m,k,n}, \tag{16}$$

$$t_{\mathrm{ave},m+1,k} = T_{m+1,k} - \sum_{n=1}^{\infty} \frac{2\sin^2\mu_n}{\mu_n(\mu_n + \sin\mu_n \cos\mu_n)} L'_{m,k,n}, \tag{17}$$

where

$$L'_{m,k,n} = \exp\{-[\mu_n^2 k\Delta \mathrm{Fo} + (m+1)\Delta \mathrm{St}]\} - L_{m,k,n}.$$

The calculations were performed to different accuracies: by summing five terms of the series, and by retaining only the first term.

A comparison of the results of the calculation of the temperature of the gas presented in Table 1 shows that the differences between the two solutions in the practically important range of Fo, Bi, and St values are admissible for engineering calculations.

It is of interest to obtain an approximate analytic solution. By retaining only the first term of the series in the general solution of the heat-conduction equation (7) for $\varphi(\mathrm{Fo}) = 1$ and $f(r, \mathrm{St}) = 0$ Eq. (9) takes the form

$$\frac{d\bar{T}}{d\mathrm{St}} + A(\mu_1)u(1, \mu_1)\frac{p}{p+\mu_1^2}\bar{T} = 0, \tag{18}$$

from which

$$\bar{T} = \frac{1}{p}\exp\left(-\mathrm{St}'\frac{p}{p+\mu_1^2}\right), \text{ where } \mathrm{St}' = A(\mu_1)u(1, \mu_1)\mathrm{St}.$$

We reduce the last expression to a form conenient for finding the inverse transform from tables [5]:

$$\bar{T} = \left[\frac{1}{p+\mu_1^2}\exp\left(\mathrm{St}'\frac{\mu_1^2}{p+\mu_1^2}\right) + \frac{\mu_1^2}{p(p+\mu_1^2)}\exp\left(\mathrm{St}'\frac{\mu_1^2}{p+\mu_1^2}\right)\right]\exp(-\mathrm{St}'). \tag{19}$$

As a result we obtain

$$T = \exp[-(\mathrm{St}' + \mathrm{Fo}')]I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'}) + \int_0^{\mathrm{Fo}'}\exp[-(\mathrm{St}' + \mathrm{Fo}')]I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'})d\mathrm{Fo}', \tag{20}$$

where $\mathrm{Fo}' = \mu_1^2\mathrm{Fo}$.

We use the property of a fundamental function [6]

$$\exp[-(x+y)]I_0(2\sqrt{xy}) = 1 - \int_0^x \exp[-(x+y)]I_0(2\sqrt{xy})dx - \int_0^y \exp[-(x+y)]I_0(2\sqrt{xy})dy,$$

to reduce the solution (20) to the following final form:

$$T(\mathrm{Fo}, \mathrm{St}) = 1 - \int_0^{\mathrm{St}'}\exp[-(\mathrm{St}' + \mathrm{Fo}')]I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'})d\mathrm{St}'. \tag{21}$$

Equation (21) is identical in form with the solution for a layer of bodies of perfect thermal conductivity [6]. The only difference is in the structure of the dimensionless arguments which in the present case depends on the shape of the bodies and the Biot number. Taking this into account the temperature of the gas can be determined from the graph of the fundamental function which ordinarily is used to calculate the heating of a layer of perfectly conducting bodies, (e.g. in [3]).

From (20) we find the rate of change of the temperature of the gas

$$\frac{\partial T}{\partial \mathrm{Fo}} = \exp[-(\mathrm{St}' + \mathrm{Fo}')] I_1(2\sqrt{\mathrm{St}'\mathrm{Fo}'}) \sqrt{\frac{\mathrm{St}'\mu_1^2}{\mathrm{Fo}}} \tag{22}$$

and substituting (21) and (22) into (7) we obtain an expression for the temperature of the material

$$t(r, \mathrm{Fo}, \mathrm{St}) = 1 - \int_0^{\mathrm{St}'} \exp[-(\mathrm{St}' + \mathrm{Fo}')] I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'}) d\mathrm{St}'$$
$$- A(\mu_1) u(r, \mu_1) \exp[-(\mathrm{St}' + \mathrm{Fo}')] I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'}). \tag{23}$$

For St' = 0 solution (23) goes over into the familiar solution of the heat-conduction equation for a constant temperature of the medium in the domain of regular behavior

$$t = 1 - A(\mu_1) u(r, \mu_1) \exp(-\mathrm{Fo}').$$

The approximate formulas (21) and (23) permit convenient rapid calculations of the heating (cooling) of a layer of bodies of various shapes for any values of Bi using a graph of the fundamental function and tables of exponential and Bessel functions.

For example, when applied to a layer of plane bodies Eq. (23) leads to the following simple computational formulas:

$$t_{\mathrm{sur}} = T - \frac{2\sin\mu_1\cos\mu_1}{\mu_1 + \sin\mu_1\cos\mu_1} \exp[-(\mathrm{St}' + \mathrm{Fo}')] I_0(2\sqrt{\mathrm{St}'\mathrm{Fo}'}), \tag{24}$$

$$t_{\mathrm{ax}} = T - \frac{1}{\cos\mu_1}(T - t_{\mathrm{sur}}), \tag{25}$$

$$t_{\mathrm{ave}} = T - \frac{\mathrm{tg}\,\mu_1}{\mu_1}(T - t_{\mathrm{sur}}). \tag{26}$$

Calculation of the heating of a plate by a stream of gas using Eqs. (21), (24)-(26) gave the same result as a computer calculation using algorithm (14)-(17) and retaining only the first term of the series.

In conclusion we present an approximate solution for boundary conditions (5) and (6):

$$T = \left\{\varphi(\mathrm{Fo}) + \int_0^{\mathrm{Fo}} \varphi(\mathrm{Fo} - \omega)\exp(-\mu_1^2\omega)\sqrt{\frac{\mathrm{St}'\mu_1^2}{\omega}} I_1(2\sqrt{\mathrm{St}'\mu_1^2\omega})\, d\omega\right\}\exp(-\mathrm{St}') + \int_0^{\mathrm{St}}\left[\int_0^1 A'(\mu_1) u'(r, \mu_1, \zeta)\, dr\right]$$
$$\times \exp[-A(\mu_1)u(1, \mu_1)(\mathrm{St} - \zeta) - \mathrm{Fo}'] I_0[2\sqrt{A(\mu_1)u(1, \mu_1)(\mathrm{St} - \zeta) - \mathrm{Fo}'}]\, d\zeta. \tag{27}$$

NOTATION

t is the temperature of the material; T is the temperature of the gas; r is the dimensionless coordinate along the thickness of the body; α is the coefficient of heat transfer from gas to material; F is the area of the heating surface from the front of the layer to any cross section perpendicular to the gas flow; Vc is the water equivalent of the gas flow; I_0 and I_1 are Bessel functions of the first kind of imaginary argument; k is the ordinal number of the time interval; m is the ordinal number of an element along the length of the layer; t_{sur} is the surface temperature of the body; t_{ax} is the temperature of the body along its axis; t_{ave} is the temperature of the body averaged over its mass; μ_1 is the first root of the characteristic equation; Fo is the Fourier number; Bi is the Biot number; St is the Stanton number.

LITERATURE CITED

1. G. A. Aksel'rud, Inzh.-Fiz. Zh., 11, 93 (1966).
2. R. F. Shklyar, R. P. Chakina, and V. M. Malkin, Sbornik nauchnykh trudov VNIIMT, Metallurgiya, No. 13 (1967), p. 104.
3. S. I. Kitaev, Yu. G. Yaroshenko, and B. L. Lazarev, Heat Transfer in a Blast Furnace [in Russian], Metallurgiya, Moscow (1968).
4. É. M. Gol'dfarb, Heat Engineering of Metallurgical Processes [in Russian], Metallurgiya, Moscow (1967).
5. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
6. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Oxford, New York (1947).